Some first thoughts on the stability of the asynchronous systems

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Abstract. The (non-initialized, non-deterministic) asynchronous systems (in the input-output sense) are multi-valued functions from m-dimensional signals to sets of n-dimensional signals, the concept being inspired by the modeling of the asynchronous circuits. Our purpose is to state the problem of the their stability.
Keywords: signal, asynchronous system, stability.

1 Introduction

$\mathbf{B} = \{0, 1\}$ is the binary Boole algebra. The function $x : \mathbf{R} \to \mathbf{B}^n$ has a limit when $t \to \infty$ if

$$\exists t_f, \forall t \geq t_f, x(t) = x(t_f) \tag{1.1}$$

The usual notation is $x(t_f) = \lim\limits_{t \to \infty} x(t)$. x is called (n-dimensional) signal if it is of the form

$$x(t) = x(t_0 - 0) \cdot \varphi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \varphi_{[t_0, t_1)}(t) \oplus x(t_1) \cdot \varphi_{[t_1, t_2)}(t) \oplus ... \tag{1.2}$$

where $t \in \mathbf{R}$. In (1.2) $\varphi_{(\ \ )} : \mathbf{R} \to \mathbf{B}$ is the characteristic function and $t_0 < t_1 < t_2 < ...$ is some unbounded sequence. We note

$$S^{(n)} = \{x | x : \mathbf{R} \to \mathbf{B}^n, x \ is \ signal\}$$

$$P^*(S^{(n)}) = \{X | X \subset S^{(n)}, X \neq \emptyset\}$$

$$S_c^{(n)} = \{x | x \in S^{(n)}, \exists \lim_{t \to \infty} x(t)\}$$

For the Boolean function $F:\mathbf{B}^m\rightarrow\mathbf{B}^n$ we note also

$$S_{F,c}^{(m)}=\{u|u\in S^{(m)},\exists\lim_{t\rightarrow\infty}F(u(t))\}$$

Any signal x has an initial time instant t_0, from the definition (1.2). It is not unique and it is precised by the condition $\forall t<t_0,x(t)=x(t_0-0)$, where (the unique) $x(t_0-0)$ is the initial value of x. In particular the constant function x satisfies the property that any t_0 is an initial time instant and x coincides with its initial value. There exist signals without final time instant t_f and respectively without final value $\lim\limits_{t\rightarrow\infty}x(t)$. If t_f exists, it is not unique and any $t_f'>t_f$ is a final time instant too. In particular, the constant function x satisfies the property that any t_f is a final time instant and x coincides with its final value.

When x is the state of a system, the problem of the existence of t_f, thus of the limit $\lim\limits_{t\rightarrow\infty}x(t)$ is the stability problem of that system.

2 Asynchronous systems

Definition We call (non-initialized, non-deterministic) asynchronous system (in the input-output sense) a function $f:U\rightarrow P^*(S^{(n)})$, where $U\in P^*(S^{(m)})$. The elements $u\in U$, respectively $x\in f(u)$ are called (admissible) inputs, respectively (possible) states, or outputs.

Remark The concept of asynchronous system has its origin in the modeling of the asynchronous circuits, where the multivalued association between the cause u and the effects $x\in f(u)$ is motivated by the changes in power supply, temperature, by the technologycal dispersion, by the errors of the measurement instruments etc.

Definition The system $g:V\rightarrow P^*(S^{(n)})$, $V\in P^*(S^{(m)})$ is called a subsystem of f if

$$V\subset U\quad and\quad \forall u\in V,g(u)\subset f(u)$$

Defnition The system $f^*:U^*\rightarrow P^*(S^{(n)})$, $U^*\in P^*(S^{(m)})$ is called the dual system of f if $U^*=\{\overline{u}|u\in U\}$ and $\forall u\in U,f^*(\overline{u})=\{\overline{x}|x\in f(u)\}$. We have noted with $\overline{u},\overline{x}$ the coordinatewise complements of these signals, for example $\overline{u}(t)=(\overline{u_1(t)},...,\overline{u_m(t)})$.

Definition We suppose that $U\cap V\neq\emptyset$ and that $\forall u\in U\cap V,f(u)\cap g(u)\neq\emptyset$. The system $f\cap g:U\cap V\rightarrow P^*(S^{(n)})$ is defined by

$$\forall u\in U\cap V,(f\cap g)(u)=f(u)\cap g(u)$$

Definition The system $f \cup g : U \cup V \to P^*(S^{(n)})$ is defined in the next manner

$$\forall u \in U \cup V, (f \cup g)(u) = \begin{cases} f(u), if \quad u \in U - V \\ g(u), if \quad u \in V - U \\ f(u) \cup g(u), if \quad u \in U \cap V \end{cases}$$

Definition Let the system $f' : U' \to P^*(S^{(n')}), U' \in P^*(S^{(m)})$. If $U \cap U' \neq \emptyset$, the parallel connection of f and f' is the system $(f, f') : U \cap U' \to P^*(S^{(n+n')})$ defined by

$$\forall u \in U \cap U', (f, f')(u) = \{z | z \in S^{(n+n')},$$

$$\forall i \in \{1, ..., n + n'\}, z_i = \begin{cases} x_i, \quad if \quad i \in \{1, ..., n\}, x \in f(u) \\ y_{i-n}, if \quad i \in \{n + 1, ..., n + n'\}, y \in f'(u) \end{cases} \}$$

Definition The system $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ is given, so that $\forall u \in U, f(u) \cap X \neq \emptyset$. The serial connection of h and f is the system $h \circ f : U \to P^*(S^{(p)})$ that is defined by

$$\forall u \in U, (h \circ f)(u) = \{y | \exists x \in f(u) \cap X, y \in h(x)\}$$

Definition The system f is called non-anticipatory, or causal if

$$\forall t_1 \in \mathbf{R}, \forall u \in U, \forall v \in U, u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)} \Longrightarrow$$

$$\Longrightarrow \{x_{|(-\infty,t_1)} | x \in f(u)\} = \{y_{|(-\infty,t_1)} | y \in f(v)\}$$

Definition The system f is initialized if

$$\exists w^0 \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = w^0$$

If so, the unique vector w^0 satisfying the previous property is called the initial state of f.

3 Steady values of the states

Definition Let the system $f : U \to P^*(S^{(n)})$, $U \subset S^{(m)}$. If

$$\exists u \in U, \exists x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

then the binary vector w is called the steady value, or the final value, or the limit when $t \to \infty$ of the state $x \in f(u)$. In the special case when

$$\exists u \in U, \exists x \in f(u), \exists w \in \mathbf{B}^n, \forall t \in \mathbf{R}, x(t) = w$$

is true, w is called a point of equilibrium of f.

Remark For any u and any $x \in f(u)$, if $w = \lim_{t\to\infty} x(t)$ exists, then it is unique.

Notation For $u \in U$, we note

$$\Sigma_f(u) = \{w | \exists x \in f(u), w = \lim_{t\to\infty} x(t)\}$$

4 Initial time and final time

Definition We say that the system f has an initial time (instant) t_0 which is

a) unbounded if

$$\forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = x(t_0 - 0)$$

b) bounded if

$$\forall u \in U, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = x(t_0 - 0)$$

c) fix (or universal) if

$$\exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t < t_0, x(t) = x(t_0 - 0)$$

We say that the system f has a final time (instant) t_f which is

a') unbounded if

$$\forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f)$$

b') bounded if

$$\forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f)$$

c') fix (or universal) if

$$\exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f)$$

Remarks There are $3 \times 3 = 9$ possibilities of combining the initial time and the final time for a system.

The next implications are true:

$$t_0 \quad fix \Longrightarrow t_0 \quad bounded \Longrightarrow t_0 \quad unbounded$$

and the next implications are true also:

$$t_f \quad fix \Longrightarrow t_f \quad bounded \Longrightarrow t_f \quad unbounded$$

5 Absolute stability

Definition a) A system f that satisfies

$$\forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called absolutely stable.

b) If

$$\forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

then f is called absolutely race-free stable, or absolutely delay-insensitive.

c) We say that f is absolutely constantly stable if it satisfies

$$\exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

Remarks The next implications are true:

$$f \quad abs \quad const \quad stable \Longrightarrow f \quad abs \quad race-free \quad stable \Longrightarrow f \quad abs \quad stable$$

On the other hand if f is absolutely stable, then it defines the system $\lim f : U \to P^*(S^{(n)})$ by $\forall u \in U, \lim f(u) = \Sigma_f(u)$ and we have identified the binary vector with the constant vector function. In case of absolute race-free stability, this system is deterministic, i.e. $\forall u \in U$, the set $\lim f(u)$ has exactly one element. If the absolute constant stability of f is true also, then $\lim f$ is the constant univalued function.

Sometimes it will be useful to write the absolute stability condition under the form

$$\forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t-0) = w$$

and similarly for the other two cases, showing the fact that x has reached its final value w sometime before t_f.

Theorem We suppose that $f : U \to P^*(S^{(n)})$, $U \subset S^{(m)}$ is an absolutely stable (an absolutely race-free stable, an absolutely constantly stable) system and let the systems $g : V \to P^*(S^{(n)})$, $V \subset S^{(m)}$, $f' : U' \to P^*(S^{(n')})$, $U' \subset S^{(m)}$. The next statements are true:

a) If $g \subset f$, then g is absolutely stable (absolutely race-free stable, absolutely constantly stable)

b) f^* is absolutely stable (absolutely race-free stable, absolutely constantly stable)

c) If $U \cap V \neq \emptyset$ and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then $f \cap g$ is absolutely stable (absolutely race-free stable, absolutely constantly stable)

d) If g is absolutely stable (absolutely race-free stable, absolutely constantly stable), then $f \cup g$ is absolutely stable (absolutely race-free stable, absolutely constantly stable)

e) If f' is absolutely stable (absolutely race-free stable, absolutely constantly stable) and if $U \cap U' \neq \emptyset$, then (f, f') is absolutely stable (absolutely race-free stable, absolutely constantly stable)

Theorem Let the systems f and $h : X \to P^*(S^{(p)})$, $X \subset S^{(n)}$. We suppose that $\forall u \in U, f(u) \cap X \neq \emptyset$; then if h is absolutely stable (absolutely constantly stable), we have that $h \circ f$ is absolutely stable (absolutely constantly stable).

Remark The statement of the previous theorem is false in the case of absolute race-free stability, in general.

Theorem The next properties are equivalent for the system f:

a) absolute stability with unbounded final time:

$$\left\{\begin{array}{l} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f) \end{array}\right. \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute)

b) absolute stability with bounded final time:

$$\left\{\begin{array}{l} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{array}\right. \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \forall t \geq t_f, x(t) = w$$

c) absolute stability with fix final time:

$$\left\{\begin{array}{l} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{array}\right. \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \forall t \geq t_f, x(t) = w$$

d) absolute race-free stability with unbounded final time:

$$\left\{\begin{array}{l} \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f) \end{array}\right. \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

e) absolute race-free stability with bounded final time:

$$\left\{\begin{array}{l} \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{array}\right. \Longleftrightarrow$$

$$\Longleftrightarrow \forall u\in U,\exists w\in \mathbf{B}^n,\exists t_f\in \mathbf{R},\forall x\in f(u),\forall t\geq t_f,x(t)=w$$

where w and t_f depend on u only (thus $\exists w,\exists t_f$ commute)

f) absolute race-free stability with fix final time:

$$\begin{cases} \forall u\in U,\exists w\in \mathbf{B}^n,\forall x\in f(u),\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=w \\ \exists t_f\in \mathbf{R},\forall u\in U,\forall x\in f(u)\cap S_c^{(n)},\forall t\geq t_f,x(t)=x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_f\in \mathbf{R},\forall u\in U,\exists w\in \mathbf{B}^n,\forall x\in f(u),\forall t\geq t_f,x(t)=w$$

g) absolute constant stability with unbounded final time:

$$\begin{cases} \exists w\in \mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=w \\ \forall u\in U,\forall x\in f(u)\cap S_c^{(n)},\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w\in \mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=w$$

h) absolute constant stability with bounded final time:

$$\begin{cases} \exists w\in \mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=w \\ \forall u\in U,\exists t_f\in \mathbf{R},\forall x\in f(u)\cap S_c^{(n)},\forall t\geq t_f,x(t)=x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w\in \mathbf{B}^n,\forall u\in U,\exists t_f\in \mathbf{R},\forall x\in f(u),\forall t\geq t_f,x(t)=w$$

i) absolute constant stability with fix final time:

$$\begin{cases} \exists w\in \mathbf{B}^n,\forall u\in U,\forall x\in f(u),\exists t_f\in \mathbf{R},\forall t\geq t_f,x(t)=w \\ \exists t_f\in \mathbf{R},\forall u\in U,\forall x\in f(u)\cap S_c^{(n)},\forall t\geq t_f,x(t)=x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w\in \mathbf{B}^n,\exists t_f\in \mathbf{R},\forall u\in U,\forall x\in f(u),\forall t\geq t_f,x(t)=w$$

where w and t_f are constant (thus $\exists w,\exists t_f$ commute)

Theorem Let the system f having the property that it is non-anticipatory and with fix final time.

a) If f is absolutely stable, then the set $\Sigma_f(u)$ depends on the restriction $u_{|(-\infty,t_f]}$ only.

b) In the case that f is absolutely delay-insensitive, the limit $\lim\limits_{t\to\infty} x(t)$ that is the same for all $x\in f(u)$ depends on $u_{|(-\infty,t_f]}$ only.

c) If f is absolutely constantly stable, $\lim\limits_{t\to\infty} x(t)$ is the same for all $x\in f(u)$ and all $u\in U$.

6 Relative stability

Definition a) A system f that satisfies

$$\forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called relatively stable.

b) If the next property is true

$$\forall u \in U \cap S_c^{(m)}, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

then f is called relatively race-free stable, or relatively delay-insensitive.

c) f is relatively constantly stable if

$$\exists w \in \mathbf{B}^n, \forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

If $U \cap S_c^{(m)} = \emptyset$ we say that the previous stability properties are trivially fulfilled and if $U \cap S_c^{(m)} \neq \emptyset$ that they are non-trivially fulfilled.

Remark Relative stability is analized similarly with the absolute stability.

7 Stability relative to a function

Definition Let the Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$.

a) A system f satisfying

$$\forall u \in U \cap S_{F,c}^{(m)}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called $F-$relatively stable (or stable relative to the function F).

b) If the next property holds

$$\forall u \in U \cap S_{F,c}^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = F(u(t_f))$$

then f is called $F-$relatively race-free stable, or $F-$ relatively delay-insensitive (race-free stable relative to the funcion F, delay-insensitive relative to the function F).

c) f is $F-$relatively constantly stable if it is $F-$relatively race-free stable and the function F is constant:

$$\exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

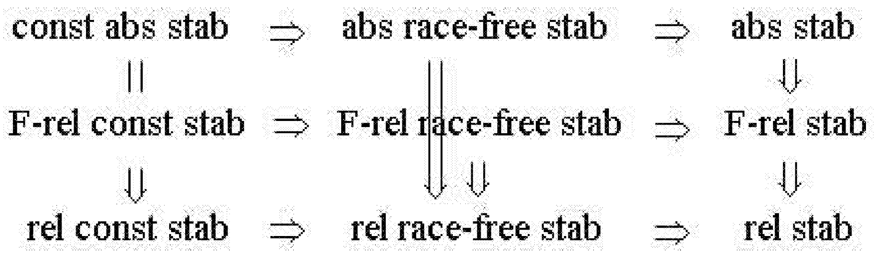

Figure 1:

If $U \cap S_{F,c}^{(m)} = \emptyset$ the previous stability properties are trivial and if $U \cap S_{F,c}^{(m)} \neq \emptyset$ they are non-trivial.

Remarks The stability of a system relative to a Boolean function is similar with the other notions of stability. We observe that the notions of $F-$relative constant stability and respectively of absolutely constant stability coincide, being at the same time a special case of $F-$relative race-free stability.

We give in Figure 1 the existing connection between the nine types of stability that were previously defined.

8 Synchronous-like, monotonous and hazard-free transitions. The fundamental mode

Definition For $x \in S^{(n)}$ and the time instances $t' < t"$, the couple $(x(t'), x(t"))$ is called transition; we say that x has a transition in the interval $[t', t"]$ from the value $x(t')$ to the value $x(t")$.

Definition By the transition $(x(t'-0), x(t"-0))$ it is understood any of the transitions $(x(t'-\varepsilon), x(t"-\varepsilon))$, where $\varepsilon > 0$ is taken sufficiently small so that

$$\forall \xi \in (0, \varepsilon], x(t'-\xi) = x(t'-0)$$

$$\forall \xi \in (0, \varepsilon], x(t"-\xi) = x(t"-0)$$

The interval on which this transition takes place is by definition any of the intervals $[t'-\xi, t"-\xi]$ with $\xi \in (0, \varepsilon]$.

Notations The usual notations for the transitions $(x(t'), x(t"))$ and $(x(t'-0), x(t"-0))$ are $x(t') \to x(t")$ and respectively $x(t'-0) \to x(t"-0)$. The interval on which $x(t'-0) \to x(t"-0)$ takes place is noted $[t'-0, t"-0]$.

Definition The next data is given: the system f, the input $u \in U$, the state $x \in f(u)$ and the instants $t' < t"$. In this case the transition

$x(t') \rightarrow x(t")$ is also called transfer of x under the input u in the interval $[t', t"]$ from the value $x(t')$ to the value $x(t")$ and we say that f transfers x under the input u (it $u-$transfers x) in the interval $[t', t"]$ from $x(t')$ to $x(t")$.

Similarly for the transition $x(t'-0) \rightarrow x(t"-0)$.

Definition a) We suppose that $w, w' \in \mathbf{B}^n, t_0, t_f \in \mathbf{R}$ and $u \in U$ exist so that

a.i) $\forall x \in f(u), \forall t < t_0, x(t) = w$

a.ii) $\forall x \in f(u), \forall t \geq t_f, x(t-0) = w'$

a.iii) $t_0 < t_f$

Then $x(t_0 - 0) \rightarrow x(t_f - 0)$ is a synchronous-like transition (or transfer); we say that f transfers synchronous-likely (any) x under the input u in the interval $[t_0 - 0, t_f - 0]$ from the value w to the value w'.

b) We suppose that $w, w' \in \mathbf{B}^n, t_f, t'_f \in \mathbf{R}$ and $u, v \in U$ exist so that

b.i) $\forall x \in f(u), \forall t \geq t_f, x(t-0) = w$

b.ii) $\forall y \in f(v), \forall t \geq t'_f, y(t-0) = w'$

b.iii) $t_f < t'_f$

b.iv) $u_{|(-\infty,t_f)} = v_{|(-\infty,t_f)}$

b.v) $\{x_{|(-\infty,t_f)} | x \in f(u)\} = \{y_{|(-\infty,t_f)} | y \in f(v)\}$

If they are true, then $y(t_f - 0) \rightarrow y(t'_f - 0)$ is a synchronous-like transition (or transfer). We also say that the system f transfers synchronous-likely (any) y under the input $v = u \cdot \varphi_{(-\infty,t_f)} \oplus v \cdot \varphi_{[t_f,\infty)}$ in the interval $[t_f - 0, t'_f - 0]$ from the value w to the value w'.

c) All the synchronous-like transitions are these from a) and b).

Remarks The attribute 'synchronous-like' given to a transition $y(t_f - 0) \rightarrow y(t'_f - 0)$ implies the fact that $y(t_f - 0) = x(t_f - 0)$ is a steady value of $x \in f(u)$ and $y(t'_f - 0)$ is a steady value $y \in f(v)$. The initial value is the same for all $x \in f(u)$ and all $y \in f(v)$ and it is treated as a steady value. Things happen as if the unique state y switches with all the coordinates simultaneously (synchronously), in discrete time, in the manner $y(k) = w, y(k+1) = w', ...$ On the other hand, the 'composition' of the synchronous-like transitions is a synchronous-like transition: if $t_f < t'_f < t"_f$ and if $y(t_f - 0) \rightarrow y(t'_f - 0)$, $y(t'_f - 0) \rightarrow y(t"_f - 0)$ are synchronous-like transitions, then $y(t_f - 0) \rightarrow y(t"_f - 0)$ is synchronous-like too.

Definition Let the system f and the input u having the property of existence of an unbounded sequence $t_0 < t_1 < t_2 < ...$ so that $x(t_k - 0) \rightarrow x(t_{k+1} - 0)$ be synchronous-like for all $k \in \mathbf{N}$ and all $x \in f(u)$. We say that f is, under the input u, in the fundamental (operating) mode.

Definition The non-empty set $U \subset S^{(m)}$ is called $\sigma-$closed if for any sequence $u^k \in U, k \in \mathbf{N}$ of inputs and any unbounded sequence $t_0 < t_1 < t_2 < ...$ of real numbers we have $u^0 \cdot \varphi_{(-\infty,t_0)} \oplus u^1 \cdot \varphi_{[t_0,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus ... \in U$.

Theorem We suppose that U is $\sigma-$closed and that f satisfies

a) it is non-anticipatory

b) it satisfies the next property of initialization with bounded initial time:

$$\forall u \in U, \exists w^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = w^0$$

where w^0 and t_0 depend on u only (thus $\exists w^0, \exists t_0$ commute)

c) it is absolutely race-free stable with bounded final time, i.e.

$$\forall u \in U, \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t-0) = w$$

where w and t_f depend on u only (thus $\exists w, \exists t_f$ commute)

Then for any sequence $u^k \in U, k \in \mathbf{N}$ of inputs, the unbounded sequence $t_0 < t_1 < t_2 < ...$ of real numbers exists so that the transitions $x(t_k - 0) \rightarrow x(t_{k+1} - 0)$, $k \in \mathbf{N}$, $x \in f(u)$ are synchronous-like, where $u \in U$ is given by

$$u = u^0 \cdot \varphi_{(-\infty,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus u^3 \cdot \varphi_{[t_2,t_3)} \oplus ...$$

Remark The previous theorem has two variants when 'f is absolutely race-free stable' is replaced by 'f is relatively race-free stable', respectively by 'f is $F-$relatively race-free stable'.

Theorem Let the system $f : U \rightarrow P^*(S^{(n)})$, with U $\sigma-$closed and we make the next suppositions:

a) f is non-anticipatory

b) it is initialized with fix initial time, i.e.

$$\exists w^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t < t_0, x(t) = w^0$$

where w^0 and t_0 are constant (thus $\exists w^0, \exists t_0$ commute)

c) the next controllability properties hold:

$$\forall w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \exists u \in U, \forall x \in f(u), \forall t \geq t_f, x(t-0) = w \qquad (8.1)$$

$$\exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \exists u \in U, \forall x \in f(u), \forall t \geq t_f, x(t-0) = w \Longrightarrow \qquad (8.2)$$

$$\Longrightarrow \forall w' \in \mathbf{B}^n, \exists t'_f \in \mathbf{R}, \exists v \in U, \forall x \in f(u \cdot \varphi_{(-\infty,t_f)} \oplus v \cdot \varphi_{[t_f,\infty)}),$$

$$\forall t \geq t'_f, x(t-0) = w'$$

Then for any sequence $w^k \in \mathbf{B}^n, k \geq 1$ of binary vectors, an unbounded sequence $t_0 < t_1 < t_2 < ...$ of real numbers and a sequence of inputs $u^k \in U, k \in \mathbf{N}$ exist so that the input $u \in U$ defined by

$$u = u^0 \cdot \varphi_{(-\infty,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus u^3 \cdot \varphi_{[t_2,t_3)} \oplus ...$$

satisfies the property

$$x(t_k - 0) = w^k$$

$$x(t_k - 0) \rightarrow x(t_{k+1} - 0) \quad are \quad synchronous - like$$

for all $k \in \mathbf{N}$ and all $x \in f(u)$.

Definition The transition $x(t') \rightarrow x(t")$ is called monotonous, if all the coordinate functions $x_i, i = \overline{1, n}$ restricted to the interval $[t', t"]$ are monotonous, i.e. they have on $[t', t"]$ at most one discontinuity point. The transition $x(t' - 0) \rightarrow x(t" - 0)$ is monotonous if all the coordinate functions $x_i, i = \overline{1, n}$ restricted to all the intervals $[t' - \varepsilon, t" - \varepsilon]$ with $\varepsilon > 0$ chosen sufficiently small are monotonous.

Definition If for $u \in U$ and $t_f < t'_f$ the transfer $x(t_f - 0) \rightarrow x(t'_f - 0)$ is synchronous-like and monotonous, $x \in f(u)$ then it is called hazard-free.

9 Conclusions

The asynchronous systems are a mathematical concept that is inspired by the modeling of the asynchronous circuits and the purpose of this paper is that of stating the stability problem for them. We can furthermore connect with this topic the notions of controllability and accessibility (by analogy we can adopt from [1] about eight definitions of controllability and four definitions of accessibility, but there exist also different points of view in the literature) we can change / replace the non-anticipation condition with other similar or dual conditions, we can suppose that f is generated by a generator function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \rightarrow \mathbf{B}^n$, that it satisfies supplementary inertial properties etc.

References

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